(7)

6/30/2019 II

1

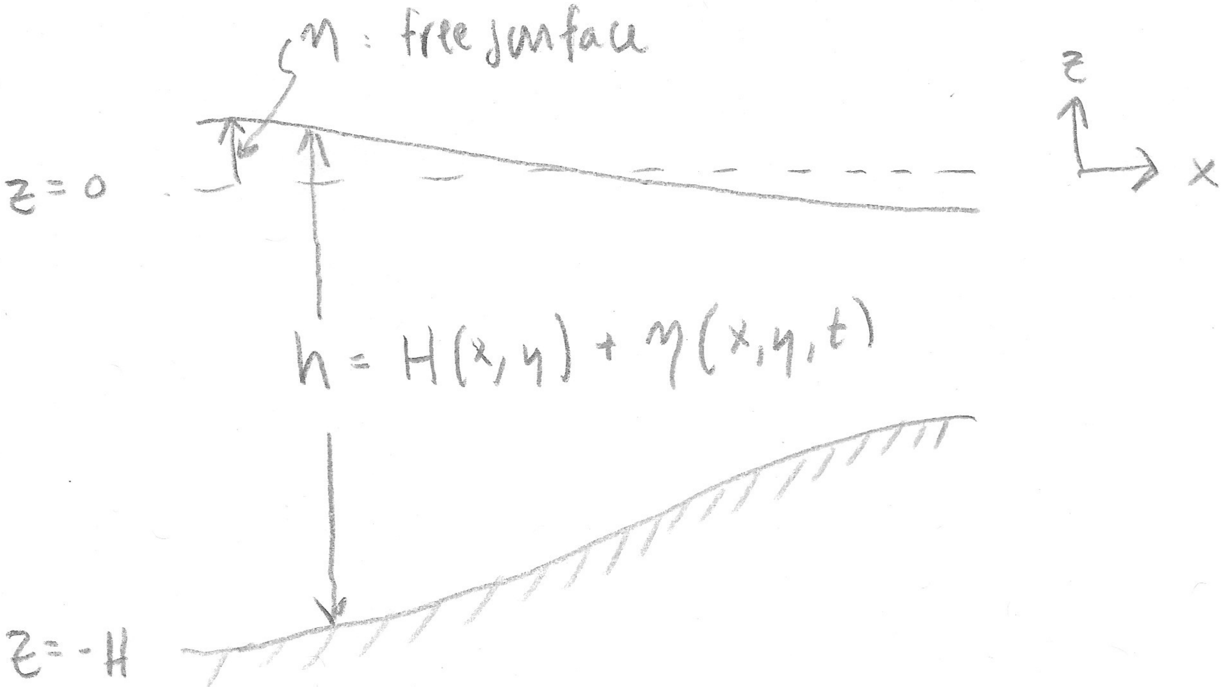
(1 3/4 h)

Frictional "Shallow Water" Equations

Useful for tides and other barotropic waves.

- Formed by taking vertical average of eqns.

define $\overline{(\ )} = \frac{1}{h}\int_{-H}^{\eta} (\ )\,dz$



- and assume ρ = const. = ρ_0

- start with vertical integral of $\boxed{\text{mass}}$

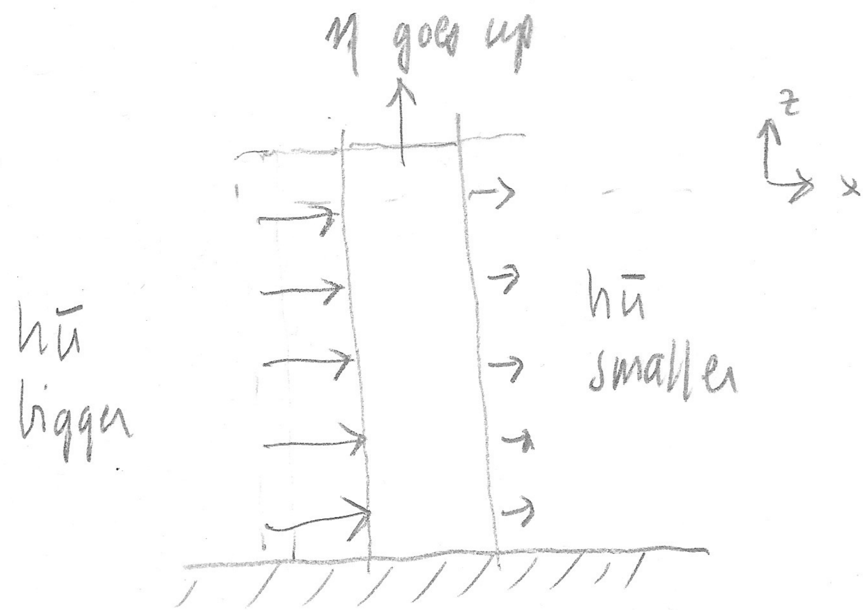
Conceptually:

$\boxed{\text{mass}}$
$$\boxed{\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(h\bar{u}) - \frac{\partial}{\partial y}(h\bar{v})}$$

rate of change of surface height = convergence of horizontal volume transport

negative of "divergence"

Example



Doing the math:

$$\int_{-H}^{\eta} \left(\nabla \cdot \underline{u} = 0 \right) dz \,, \quad \text{assume } v = 0 \text{ for simplicity}$$

$$\Rightarrow \int_{-H}^{\eta} \frac{\partial w}{\partial z}\, dz + \int_{-H}^{\eta} \frac{\partial u}{\partial x}\, dz = 0$$

$$\Rightarrow \underbrace{w\big|_{\eta} - w\big|_{-H}}_{\text{use "Kinematic b.c."}} + \underbrace{\frac{\partial}{\partial x}\int_{-H}^{\eta} u\, dz - u\big|_{\eta}\frac{\partial \eta}{\partial x} + u\big|_{-H}\frac{\partial(-H)}{\partial x}}_{\text{use Leibniz's Rule}} = 0$$

use "Kinematic b.c."

$$w\big|_{\eta} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\big|_{\eta}\frac{\partial \eta}{\partial x}$$

$$-w\big|_{-H} = -\frac{D(-H)}{Dt} = -u\big|_{-H}\frac{\partial(-H)}{\partial x}$$

after cancellation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\int_{-H}^{\eta} u\, dz = 0 \quad \text{or} \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left(h\bar{u}\right) = 0 \quad \checkmark$$

Next consider $\boxed{z \text{ mom}}$ with ρ = const.

$$\int_z^\eta \left(\frac{\partial p}{\partial z} = -\rho g \right) dz$$

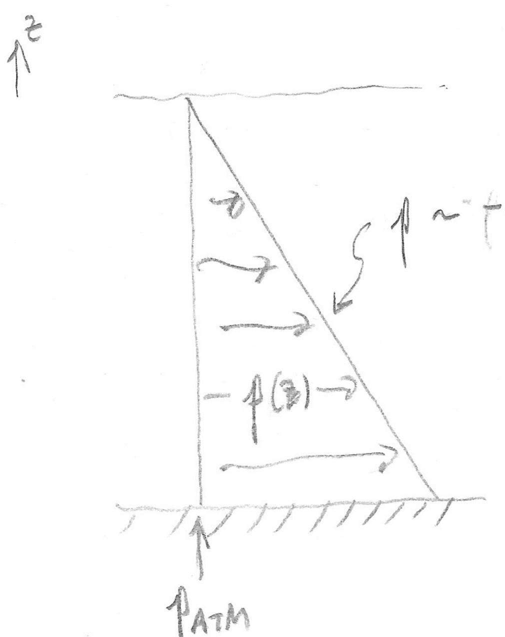
take vertical integral to find $p(z)$

$$\Rightarrow \quad p\big|_\eta - p(z) = -\rho g(\eta - z)$$

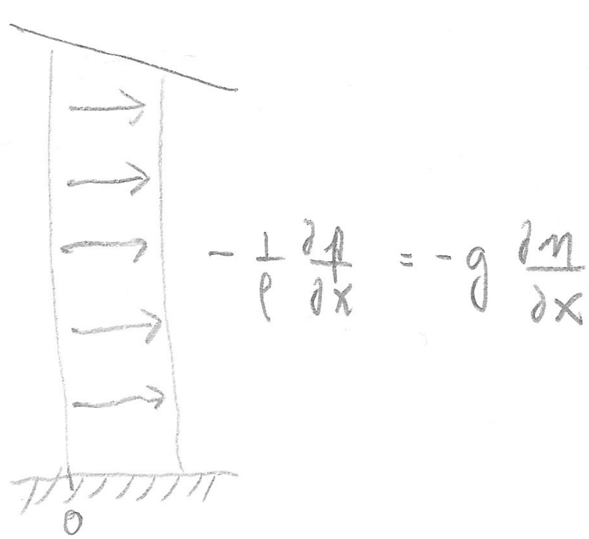
$= p_{ATM}$ assume constant

$$\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

barotropic pressure gradient : independent of depth



pressure



force due to pressure

- finally, take vertical average of $\boxed{x \text{ mom}}$

$$\overline{\frac{\partial u}{\partial t}} \rightarrow \frac{\partial \bar{u}}{\partial t}, \quad \overline{fv} \rightarrow f\bar{v} \quad \text{simple}$$

assume $\overline{u \frac{\partial u}{\partial x}} \rightarrow \bar{u}\,\bar{u}_x$ use subscripts for partial derivatives

Friction is harder

$$\frac{1}{h}\int_{-H}^{\eta} \left(A \frac{\partial u}{\partial z}\right)_z dz = \frac{1}{h}\left[\left(A\frac{\partial u}{\partial z}\right)\bigg|_{\eta} - A\left(\frac{\partial u}{\partial z}\right)\bigg|_{-H}\right]$$

bottom stress

wind stress (ignore)

Bottom stress follows a "quadratic drag law"

$$A\frac{\partial u}{\partial z}\bigg|_{-H} = C_d\sqrt{\bar{u}^2+\bar{v}^2}\,\bar{u}, \quad C_d \sim 3\times10^{-3}$$

Drag coefficient

so $\boxed{x \text{ mom}}$ is

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\,\bar{u}_x + \bar{v}\,\bar{u}_y - f\bar{v} = -g\eta_x - \frac{C_d\sqrt{\bar{u}^2+\bar{v}^2}}{h}\,\bar{u}$$

and for tidal problems we often "linearize", <u>assuming</u>

$$\frac{[\bar{u}\,\bar{u}_x]}{[\bar{u}_t]} \ll 1$$

$\Rightarrow$ spatial scales of u variation are large : <u>check later</u>

$$\frac{[\eta]}{[H]} \ll 1$$

small perturbations of SSH

$\sqrt{\bar{u}^2+\bar{v}^2} = \mathcal{U}$ typical velocity scale

H = constant. for simplicity

then defining $R \equiv \frac{C_d \mathcal{U}}{H}$ "Rayleigh drag"

we have (assume $\bar{v} = 0$)

$$\boxed{\begin{aligned} &\text{x mom} \quad \bar{u}_t = -g\eta_x - R\bar{u} \\ &\text{mass} \quad \eta_t + \underbrace{H\bar{u}_x}_{} = 0 \end{aligned}}$$

linear, frictional shallow water (SW) equations

scaling $\frac{[u h_x]}{[h u_x]} \sim \frac{[\eta]}{[H]} \ll 1$

IN-CLASS EXERCISE ☆ write these including $\bar{v}$